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The modelling of microconvection in an infinite strip☆

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ABSTRACT

A model of the microconvection of an isothermally incompressible fluid, which can be used to investigate convection in weak force fields and on microscopic scales and can be characterized by non-solenoidality of the velocity field, is considered. An invariant solution in an infinite vertical strip occupied by a fluid is studied in the case where the heat flux on the two opposite faces of the strip fluctuates in antiphase. The use of the model of microconvection to construct an invariant solution gives rise to several non-standard-value initial-boundary problems. Their solvability in classes of Holder functions is proved.

1. Model of microconvection

It has been noted[1,2] that the classical Oberbeck–Boussinesq model is unsuitable for describing convection if the microconvection parameter, which is equal to the ratio between the orders of the velocities generated by volume expansion of the fluid and the buoyancy factor, is fairly small. The term "microconvection" was introduced to describe the convection of a fluid under low gravity, as well as on microscopic scales, and for fluids, whose properties ensure small values for the microconvection parameter. The derivation of the classical equations of convection and of the Oberbeck–Boussinesq equations (see, for example, Ref. 2) from the general laws of conservation of mass, momentum and energy relied on a simplification of these laws. This simplification was based on the hypothesis that a fluid is isothermally incompressible, under which the equation of state of the medium describes a linear dependence of the density on the temperature, and on the assumption that when the motion of a fluid is described, it can be considered to be incompressible, i.e., the velocity field can be considered to be solenoidal. In the momentum conservation equation, the small deviations of the density from the average value caused by the non-uniformity of the temperature are taken into account only in the buoyancy force. The effect of the dissipative forces is not taken into account in the energy conservation equation.

If we start out from the exact mass and momentum conservation equations, but take a simplified energy conservation equation, as before, and assume that all the transfer coefficients are constant, we will obtain a model of microconvection, and the velocity field will be non-solenoidal. Assuming, however, that the dependence of the fluid density on the temperature T has the form $\rho = \rho_0/(1+\beta T)$, where β is the coefficient of thermal expansion, we can write down a model of microconvection with a new required velocity $\mathbf{W} = \mathbf{V} - \beta\chi\nabla T$, where χ is the thermal diffusivity. Then div $\mathbf{W} = 0$. The temperature dependence of the fluid density used in the deriving the model of microconvection enables us not just to switch to a solenoidal field of the modified velocity. The heat capacity at constant pressure does not depend on the pressure if and only if the equation of state has the form indicated (see Ref. 3*).

We will consider as mathematical models for modelling the convection of a fluid under weak gravity, the initial-boundary-value problem for the classical Oberbeck–Boussinesq equations of convection and for the equations of microconvection of an isothermally incompressible fluid.

The mathematical model of microconvection consists of finding the modified velocity $\mathbf{W}$, the modified pressure q (see Ref. 1 and 2 regarding the relation to the fluid pressure p) and the temperature that satisfy the system of equations

$$\mathbf{W}_t + \mathbf{W}\cdot\nabla\mathbf{W} + \beta\chi(\nabla T\cdot\nabla\mathbf{W} - \nabla\mathbf{W}\cdot\nabla T) + \beta^2\chi^2(\Delta T\cdot\nabla T - \nabla|\nabla T|^2) =$$
$$= (1+\beta T)(-\nabla q + \nu\Delta\mathbf{W}) - \beta g T, \quad \operatorname{div}\mathbf{W} = 0$$
$$T_t + \mathbf{W}\cdot\nabla T + \beta\chi|\nabla T|^2 = (1+\beta T)\chi\Delta T \qquad (1.1)$$

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in the region of flow Ω, the initial conditions at $t = 0$ and the boundary conditions on the surface Σ

$$\mathbf{W}(\mathbf{x}, 0) = \mathbf{W}_0(\mathbf{x}), \quad T = T_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{1.2}$$

$$\mathbf{W}(\mathbf{x}, t) = -\beta\chi\nabla T, \quad \frac{\partial T}{\partial n} = f(\mathbf{x}, t), \quad \int_\Sigma f(\mathbf{x}, t) d\Sigma = 0, \quad \mathbf{x} \in \Sigma, \quad t > 0 \tag{1.3}$$

Note that the condition of zero integral flow ensures the necessary condition of solvability of the problem in a closed region with rigid impenetrable boundaries. This constraint can be avoided by considering the convective motions in a region with a free or elastic boundary.[†] At the same time, the stationary problem for the equations of microconvection is correctly stated both under a temperature condition of the second kind on the boundary of the region and under a condition of the first kind.[4]

2. Statement of the problem

The search for an invariant solution of microconvection problem (1.1)–(1.3) in a vertical strip reduces to solving non-standard initial-boundary-value problems. The methods previously developed[2,5,6] enable us to prove the unique solvability of each of them under the smoothness and matching conditions imposed on the initial and boundary data.

In a Cartesian system of coordinates (x, y, z) chosen so that the vector of the force of gravity has the form $\mathbf{g} = (0, -g, 0)$, suppose a fluid fills an infinite vertical strip $|x| \le a$, on whose rigid boundaries a heat flux is specified. If the magnitude of the heat flux is independent of z, planar flows are possible in the strip. The initial velocity and temperature distributions (1.2) are independent of z, and the z component of the velocity is equal to zero. The special class of solutions of the system of microconvection equations (1.1), which are invariant under the operator $\partial/\partial y + \varphi(t)\partial/\partial q$, where $\varphi(t)$ is an arbitrary function of time, will be considered. According to the theory of the group properties of differential equations,[7] such solutions have the form

$$\mathbf{W} = (x, \upsilon, 0), \; u = u(x, t), \; \upsilon = \upsilon(x, t), \; T = T(x, t), \; q = (\varphi(t) - g)y + h(x, t) \tag{2.1}$$

It follows from the second equation of system (1.1) that $u = U(t)$, and it follows from the first equation that $U'(t) = -(1 + \beta T)h_x$. The problem of finding the function $h(x, t)$ is not fundamental, and when U and T are known, the function h is specified, apart from an additive function of time.[1,2] The search for the invariant solution (2.1) reduces to solving the second boundary-value problem for the non-linear heat conduction equation and then solving the first boundary-value problem for the linear equation, which is not a differential equation in the usual sense. Equations of this type are called loaded equations.[8] Difference methods of solving initial-boundary-value problems for loaded differential and integrodifferential equations have been investigated.[9] The special case of a problem with a loaded equation was considered in Voronin's thesis.[1]

The initial-boundary-value problems that arise during the search for the invariant solution (2.1) in a vertical strip will be considered in dimensionless form. We will select the half-width a of the strip as the characteristic dimension, a^2/ν as the characteristic time, $\varepsilon\chi/a$ as the characteristic velocity and T_* as the characteristic temperature, where ν is the kinematic viscosity. We will introduce two dimensionless parameters, viz., the Prandtl number $\mathrm{Pr} = \nu/\chi$ and the Boussinesq number $\varepsilon = \beta T_*$, and we will retain the original notations U, T and υ for the dimensionless functions.

The initial-boundary-value problems for the non-linear heat conduction equation and the y component of the velocity have the form

$$\begin{aligned} &PrT_t + \varepsilon(U + T_x)T_x = (1 + \varepsilon T)T_{xx} \\ &T(x, 0) = T_0(x), \quad |x| \le 1; \quad T_x(-1, t) = T_x(1, t) = -U(t), \quad t \ge 0 \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\upsilon_t + \frac{\varepsilon}{Pr}(U + T_x)\upsilon_x = (1 + \varepsilon T)(\upsilon_{xx} - \varphi) + \varepsilon T g \\ &\upsilon(x, 0) = \upsilon_0(x), \quad |x| \le 1; \quad \upsilon(-1, t) = \upsilon(1, t) = 0, \quad t \ge 0 \end{aligned} \tag{2.3}$$

Note that since the quantity $T_x(\pm 1, t)$ is proportional to the heat flux on the boundary, which is regarded as specified, the function $U(t)$ can also be regarded as specified in problem (2.2). The function φ satisfies the relation

$$\varphi(t) = \frac{1}{2}(\upsilon_x(1, t) - \upsilon_x(-1, t)) + \frac{\varepsilon}{2}\int_{-1}^{1} \frac{T_0}{1 + \varepsilon T_0} dx$$

We will assume that the conditions

$$U(t) \in C^{1 + \alpha/2}([0, t_f]), \quad T_0(x) \in C^{3 + \alpha}([-1, 1]), \quad \upsilon_0(x) \in C^{2 + \alpha}([-1, 1]) \tag{2.4}$$

† Yudovich VI. Convection of an isothermally incompressible fluid. Article deposited in the All-Union Institute of Scientific and Technical Information (VINITI). 28 May 1999, No. 1699-V99.

hold for an arbitrary finite t_f. We take the matching conditions in the form

$$-U(0) = T_{0x}(\pm 1)$$
$$\frac{1}{Pr}[1+\varepsilon T_0(\pm 1)]T_{0xxx}(\pm 1) + U'(0) = 0$$
$$\upsilon_0(\pm 1) = 0, \quad (1+\varepsilon T_0(\pm 1))[\upsilon_{0xx}(\pm 1) - \varphi(0)] + \varepsilon T_0(\pm 1)g = 0$$
$$\varphi(0) = \frac{1}{2}(\upsilon_{0x}(1) - \upsilon_{0x}(-1)) + \frac{\varepsilon}{2}\int_{-1}^{1}\frac{T_0}{1+\varepsilon T_0}dx \tag{2.5}$$

3. The solvability of the initial-boundary-value problems for the heat conduction equation and the loaded equation

If we seek the functions T and υ in the form of expansions in series of powers of the small parameter ε, the expansion of T in powers of ε begins from the zero-order term, and the expansion of υ begins from the first-order term:

$$T = \sum_{k=0}^{\infty}\varepsilon^k T^{(k)}, \quad \upsilon = \sum_{k=1}^{\infty}\varepsilon^k \upsilon^{(k)} \tag{3.1}$$

The leading terms in the expansions are solutions of the following problems

$$\Pr T_t^{(0)} = T_{xx}^{(0)}; \quad T^{(0)}(x,0) = T_0(x), \quad T_x^{(0)}(-1,t) = T_x^{(0)}(1,t) = -U(t) \tag{3.2}$$

$$\upsilon_t^{(1)} = \upsilon_{xx}^{(1)} - \frac{1}{2}\left[\upsilon_x^{(1)}(1,t) - \upsilon_x^{(1)}(-1,t) + \int_{-1}^{1}T_0dx\right] + gT^{(0)}$$
$$\upsilon^{(1)}(x,0) = \upsilon_0(x), \quad \upsilon^{(1)}(-1,t) = \upsilon^{(1)}(1,t) = 0 \tag{3.3}$$

with matching conditions that are corollaries of conditions (2.5) obtained by separating them for ε.

The function $T^{(0)}$ is the solution of the standard problem for the heat conduction equation (3.2). The problem of its solvability can be investigated using to a well-known approach.[5]

For the function $\upsilon^{(1)}$ the question of the solvability of problem (3.3) is complicated in the general case. This problem is non-standard because the equation contains the derivative of the function sought at boundary points. It will be investigated using an additional representation of $\upsilon^{(1)}$ in the form of an expansion in even and odd components.

We note that all the remaining functions $T^{(k)}$ and $\upsilon^{(k)}$ can be specified in a recursive form as solutions of the following boundary-value problems

$$\Pr T_t^{(k)} = T_{xx}^{(k)} - U(t)T_x^{(k-1)} + \sum_{j=0}^{k-1}[T_x^{(j)}T_{xx}^{(k-1-j)} - T_x^{(j)}T_x^{(k-1-j)}]$$
$$T^{(k)}(x,0) = 0, \quad T_x^{(k)}(-1,t) = T_x^{(k)}(1,t) = 0; \quad k \geq 1 \tag{3.4}$$

$$\upsilon_t^{(k)} = \upsilon_{xx}^{(k)} - \frac{1}{2}[\upsilon_x^{(k)}(1,t) - \upsilon_x^{(k)}(-1,t)] - \frac{1}{2}\int_{-1}^{1}(-1)^{k-1}T_0^k dx - \frac{1}{\Pr}(U + T_x^{(0)})\upsilon_x^{(k-1)} -$$
$$-\frac{1}{2}\sum_{j=0}^{k-2}T^{(j)}\upsilon_x^{(k-1-j)}\Big|_{-1}^{1} - \frac{1}{2}\sum_{j=0}^{k-1}T^{(k-1-j)}\int_{-1}^{1}(-1)^{j-1}T_0^j dx -$$
$$-\frac{1}{\Pr}\left[\sum_{j=1}^{k-2}T_x^{(j)}\upsilon_x^{(k-1-j)}\right] + \sum_{j=0}^{k-2}T^{(j)}\upsilon_{xx}^{(k-1-j)} + gT^{(k-1)}$$
$$\upsilon^{(k)}(x,0) = 0, \quad \upsilon^{(k)}(-1,t) = \upsilon^{(k)}(-1,t) = 0; \quad k \geq 2 \tag{3.5}$$

The necessary matching conditions hold here as a consequence of conditions (2.5).

3.1. The solvability of the initial-boundary-balue problems for the leading expansion terms

We will introduce concise notation for certain spaces that will be encountered frequently below:

$$\tilde{C}^m = C^{m+\alpha,(m+\alpha)/2}([-1,1]\times[0,t_f]), \quad m = 2, 3$$

When conditions (2.4) and matching conditions (2.5) hold, boundary-value problem (2.2) for the temperature is solvable, and $T^{(0)} \in \tilde{C}^3$.[5]

Next, let $\upsilon^{(1)} = \upsilon_1 + \upsilon_2$, where υ_1 is the even component, and υ_2 is the odd component of the velocity $\upsilon^{(1)}$ along x. For υ_2 we have the classical first boundary-value problem

$$\upsilon_{2t} = \upsilon_{2xx} + \bar{F}_2(x, t), \quad t \in [0, t_f], \quad x \in [-1, 1]; \quad \upsilon_2(x, 0) = \upsilon_{20}(x), \quad \upsilon_2(\pm 1, t) = 0$$

Here $\bar{F}_2(x, t)$ and $\upsilon_{20}(x)$ are odd components of the functions $\int_{-1}^{1} T_0(x)dx + gT^{(0)}(x, t)$ and $\upsilon_0(x)$, respectively.

The solvability of the problem for υ_2 can be proved using to a well-known procedure.[5] The function $\upsilon_2(x, t)$ belongs at least to the class $\tilde{C}^2$, since $\upsilon_{20}(x) \in C^{2+\alpha}([-1, 1])$ and $F_2(x, t) \in \tilde{C}^3$.

We will dwell on the problem for the even component of the velocity, in whose equation a non-local derivative remains:

$$\upsilon_{1t} = \upsilon_{1xx} - \upsilon_{1x}(1, t) + \bar{F}_1(x, t), \quad t \in [0, t_f], \quad x \in [-1, 1] \tag{3.6}$$

$$\upsilon_1(x, 0) = \upsilon_{10}(x), \quad |x| \le 1 \tag{3.7}$$

$$\upsilon_1(-1, t) = \upsilon_1(1, t) = 0, \quad t \ge 0 \tag{3.8}$$

Here $\bar{F}_1 = \bar{F}_1(x, t)$ is the even component of the function $\int_{-1}^{1} T_0(x)dx + gT^{(0)}(x, t)$ of the class $\tilde{C}^3$ in Eq. (3.3). The initial function $\upsilon_{10}(x)$ (the even component of the initial velocity) is a function of the class $C^{2+\alpha}([-1, 1])$. In particular, the equality $\upsilon_{10}(\pm 1) = 0$ holds here as a consequence of the conditions $\upsilon_0(\pm 1) = 0$ (see the corresponding condition (2.5)).

We expand the function υ_1 in a generalized Fourier series, taking its eveness into account, in such a way that boundary conditions (3.8) are automatically satisfied. These conditions have the form

$$\upsilon_1 = \sum_{n=0}^{\infty} u_n(t)\cos(\lambda_n x), \quad \lambda_n = \pi\left(n + \frac{1}{2}\right)$$

Taking into account the orthonormality of the functions $\cos\lambda_n x$ in the interval $[-1, 1]$, we write Eq. (3.6) in the form

$$\sum_{n=0}^{\infty} [u_n'(t) + \lambda_n^2 u_n(t) + e_n f(t) - F_n(t)]\cos\lambda_n x = 0 \tag{3.9}$$

where $f(t)$ is an unknown function of the values of the derivative of the function sought on the boundary $f(t) = \upsilon_{1x}(1, t)$. It follows from Eq. (3.9) that the following representation holds for any n

$$u_n(t) = C_n\exp(-\lambda_n^2 t) + \int_0^t (-e_n f(\tau) + F_n(\tau))\exp(-\lambda_n^2(t-\tau))d\tau \tag{3.10}$$

The following notation was introduced for the generalized Fourier coefficients in the expansions of unity, the right-hand side of Eq. (3.6) and the initial function (3.7) in series in $\cos\lambda_n x$

$$e_n = (-1)^n\frac{2}{\lambda_n}, \quad F_n(t) = 2\int_0^1 \bar{F}_1(x, t)\cos\lambda_n x dx, \quad C_n = 2\int_0^1 \upsilon_{10}(x)\cos\lambda_n x dx$$

The properties of the Fourier coefficients are specified by the properties of the corresponding functions for which the expansions were obtained, and will be taken into account below. We also note that

$$f(t) = \sum_{n=0}^{\infty} (-1)^{n+1}\lambda_n u_n(t), \quad f(0) = \sum_{n=0}^{\infty} (-1)^{n+1}\lambda_n C_n$$

We multiply each equality (3.10) by $(-1)^{n+1}\lambda_n$ and sum over n from 0 to ∞. Then, to find the function $f(t)$, we have the integral equation

$$f(t) = 2\int_0^t f(\tau)\sum_{n=0}^{\infty}\exp(-\lambda_n^2(t-\tau))d\tau + Q(t) \tag{3.11}$$

where

$$Q(t) = \sum_{n=0}^{\infty}\left[(-1)^{n+1}\lambda_n\left(C_n\exp(-\lambda_n^2 t) + \int_0^t F_n(\tau)\exp(-\lambda_n^2(t-\tau))d\tau\right)\right] \tag{3.12}$$

Here $Q(0)=f(0)$, and the kernel of the integral equation is a function of the form

$$K(z) = \sum_{n=0}^{\infty} \exp(-\lambda_n^2 z); \quad z = t-\tau, \quad \tau\in[0,t], \quad t\in[0,t_f] \tag{3.13}$$

It should be noted that convergence of the kernel is equivalent to convergence of the improper integral

$$\int_0^{\infty} \exp\left(-\pi^2\left(x+\frac{1}{2}\right)^2 z\right)dx = \frac{1}{2\sqrt{\pi z}} - \frac{1}{2\sqrt{\pi z}}\Phi\left(\frac{\pi\sqrt{z}}{2}\right); \quad \Phi(\alpha) = \frac{2}{\sqrt{\pi}}\int_0^{\alpha}\exp(-y^2)dy$$

Here $\Phi(\alpha)$ is Laplace's function.[10] The second term is already a smooth function when $z\geq 0$.

Representing series (3.13) by an integral of the step function

$$c(z,x) = c_n, \quad n\leq x< n+1, \quad n = 0, 1, \ldots; \quad c_n = \exp\left(-\pi^2\left(\frac{2n+1}{2}\right)^2 z\right)$$

we can obtain the limit

$$0\leq\int_0^{\infty} c(z,x)dx - \frac{1}{2\sqrt{\pi z}} + \frac{1}{2\sqrt{\pi z}}\Phi\left(\frac{\pi\sqrt{z}}{2}\right) \leq \frac{1}{\sqrt{\pi z}}\Phi\left(\frac{\pi\sqrt{z}}{2}\right)$$

which enables us to isolate a weak singularity of the form $1/\sqrt{z}$ from the kernel (3.13) and to rewrite integral equation (3.11) in the form

$$f(t) = \lambda\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}d\tau + \bar{Q}(t), \quad \bar{Q}t = \int_0^t L(t-\tau)f(\tau)d\tau + Q(t), \quad \lambda = \frac{1}{2\sqrt{\pi}} \tag{3.14}$$

Let $L(t-\tau)$ be the smooth component of the polar kernel. Equation (3.14) is classified as an Abel-type Volterra integral equation of the second kind.[11,12] Integration of the kernel component $\mathcal{K}(t,\tau)=1/\sqrt{t-\tau}$ of integral equation (3.14) leads to the expression

$$\mathcal{K}_2(t,\tau) = \int_{\tau}^{t}\frac{ds}{\sqrt{t-s}\sqrt{s-\tau}} = \pi$$

Using a combined form of the two parts of Eq. (3.14) and the function $\lambda\mathcal{K}(t,\tau)$, we obtain the Volterra integral equation

$$f(t) = \lambda^2\pi\int_0^t f(\tau)d\tau + Q_2(t); \quad Q_2(t) = \bar{Q}(t) + \lambda^2\int_0^t\frac{\bar{Q}(s)}{\sqrt{t-s}}ds \tag{3.15}$$

with a constrained kernel consisting of $\mathcal{K}_2(t,\tau)$ and the result of combining $\int_0^t L(t-\tau)f(\tau)d\tau$ and $\lambda\mathcal{K}(t,\tau)$. After solving integral equation (3.15), we obtain the solution of the original integral equation (3.14).

Note that the Fourier coefficients C_n and $F_n(t)$ of the functions $\upsilon_{10}(x)\in C^{2+\alpha}([-1,1])$ ($\upsilon_{10}(\pm1)=0$) and $\bar{F}_1(x,t)\in\tilde{C}^3$ have properties that ensure convergence of the integral in relation (3.12), since the following inequalities hold[13]

$$|C_n|\leq M_c/n^3, \quad |F_n(t)|\leq M_F/n$$

These estimates lead to a conclusion regarding the convergence of the series in relation (3.12), since for $t\geq 0$ and $x\in[-1,1]$ there are convergent majorant series for them of the forms $\sum_{n=1}^{\infty}\frac{\bar{M}_C}{n^2}$ and $\sum_{n=1}^{\infty}\frac{\bar{M}_F}{n^2}$, respectively, whence it follows that $Q(t)\in C^{2+(1+\alpha)/2}([0,t_f])$ (this is ensured by the properties of the function $\bar{F}_l$).

Thus, integral equation (3.15) is solvable.[12] The function $f(t)$ belongs at least to the class $C^{1+\alpha/2}([0,t_f])$, enabling us to assert that problem (3.6)–(3.8) is solvable in the class of functions $\tilde{C}^2$, in agreement with the existing results.[5]

3.2. *Scheme of the proof of the solvability of initial-boundary-value problems (3.4), (3.5)*

The solution of problem (2.2), (2.3), which can be represented by the formal power series (3.1), is determined successively as solutions of problems (3.2), (3.3) and (3.4), (3.5). The solvability of problem (3.2), (3.3) for the leading expansion terms, which was proved in the preceding section, leads to the limits[5]

$$|T^{(0)}|^{(3+\alpha)} \leq \mathcal{C}_1[|T_0|^{3+\alpha} + |U|^{1+\alpha/2}] = \mathcal{C}_1[C_0+C_U] = C_T$$

$$|\upsilon^{(1)}|^{(2+\alpha)} \leq \mathcal{C}_2[C_f + |\upsilon_0|^{2+\alpha} + |T_0|^{3+\alpha}] = C_\upsilon$$

The principal points in the proof of the solvability of problem (3.6)-(3.8) for the highest-order approximation $\upsilon^{(1)}$ are the search for a solution in the form of a sum of the odd and even components and the subsequent method for determining the function $f(t)$. The kernel of the integral equation for finding $f(t)$ is specified by the structure of the preceding differential equation, and the smoothness properties of the function sought depend mainly on the smoothness of the corresponding approximation for the temperature. The solvability of the recursive boundary-value problems (3.4), (3.5) in the Holder classes $\tilde{C}^m$ can be proved by induction using the method demonstrated above. Note that, according to Ref. 5 and the structure of differential equations (3.4) and (3.5), the following limits hold

$$\left|T^{(k)}\right|^{(3+\alpha)} \le \mathcal{C}_1\left\{C_U\left|T^{(k-1)}\right|^{(3+\alpha)} + 2\sum_{j=0}^{k-1}\left|T^{(j)}\right|^{(3+\alpha)}\left|T^{(k-1-j)}\right|^{(3+\alpha)}\right\}; \quad k \ge 1$$

$$\left|\upsilon^{(k)}\right|^{(2+\alpha)} \le \mathcal{C}_2\left\{C_0^k + \frac{1}{\Pr}(C_U + C_T)\left|\upsilon^{(k-1)}\right|^{(2+\alpha)} + \sum_{j=0}^{k-1}\left|T^{(k-j-1)}\right|^{(3+\alpha)}C_0^j + \right.$$
$$\left. + \left(2 + \frac{1}{\Pr}\right)\sum_{j=0}^{k-2}\left|T^{(j)}\right|^{(3+\alpha)}\left|\upsilon^{(k-j-1)}\right|^{(2+\alpha)} + g\left|T^{(k-1)}\right|^{(3+\alpha)}\right\}; \quad k \ge 2$$

We require that the initial temperature $T_0 = T_0(x)$ should satisfy the inequality

$$C_0 = \left|T_0\right|^{3+\alpha} \le 1 \tag{3.16}$$

We will demonstrate the convergence of series (3.1) for sufficiently small $\varepsilon > 0$. Along with series (3.1), consider the power series with constant coefficients

$$z = \sum_{j=0}^{\infty}\varepsilon^j z_j, \quad y = \sum_{j=1}^{\infty}\varepsilon^j y_j \tag{3.17}$$

where

$$z_0 = \left|T^{(0)}\right|^{(3+\alpha)}, \quad y_1 = \left|\upsilon^{(1)}\right|^{(2+\alpha)}$$

$$z_k = \mathcal{C}_1 C_U z_{k-1} + 2\mathcal{C}_1\sum_{j=0}^{k-1} z_j z_{k-j-1}; \quad k \ge 1$$

$$y_k = \mathcal{C}_2\left[C_0^k + \frac{1}{\Pr}(C_U + C_T)y_{k-1} + \sum_{j=0}^{k-1} z_{k-1-j}C_0^j + \left(2 + \frac{1}{\Pr}\right)\sum_{j=1}^{k-2} z_j y_{k-j-1} + g z_{k-1}\right]; \quad k \ge 2$$

By analogy with the previously described techniques,[6] we can show that the power series z converges to the small positive root $\tilde{z}$ of the quadratic equation $z = z_0 + \varepsilon(\mathcal{C}_1 C_U z + 2\mathcal{C}_1 z^2)$ if ε satisfies the condition $\varepsilon < \varepsilon_0$, and ε_0 is expressed in terms of $\mathcal{C}_1$, C_U, z_0 and is a consequence of the positive values of the discriminant and one of the coefficients of the quadratic equation. The first series z in (3.17) is majorant for the corresponding series (3.1) if $T^{(k)}$ are replaced by their norms in the space $C^{(3+\alpha)}$.

The series y converges to the solution of the equation

$$y = \varepsilon y_1 + \varepsilon^2\mathcal{C}_2\left[\frac{C_0^2}{1 - C_0} + \frac{1}{\Pr}(C_U + C_T)y + \left(2 + \frac{1}{\Pr}\right)zy + \frac{z}{1 - C_0} + gz\right]$$

if we require that the inequality $\varepsilon < \varepsilon_0$, where $\bar{\varepsilon}_0$ is expressed in terms of $\mathcal{C}_2$, C_U, C_T, Pr and $\tilde{z}$, would hold.

It should be noted that the second series y in (3.17) is also majorant for the series that is obtained from the corresponding series (3.1) if the functions $\upsilon^{(k)}$ are replaced by their norms in the space $C^{(2+\alpha)}$. Therefore, series (3.1) converge to the solution of problems (2.2) and (2.3) if and only if $\varepsilon \in [0, \tilde{\varepsilon}]$, $\bar{\varepsilon} \le \min[\varepsilon_0, \tilde{\varepsilon}_0]$.

Thus, we have the following theorem.

Theorem. *Let conditions (2.4), (2.5) and (3.16) hold. There is a $\tilde{\varepsilon} > 0$ such that for $0 \le \varepsilon \le \tilde{\varepsilon}$ problems (2.2) and (2.3) have a solution of the following form*

$$\upsilon^{(1)} \in C^{2+\alpha, 1+\alpha/2}([-1, 1] \times [0, t_f]), \quad T^{(0)} \in C^{3+\alpha, (3+\alpha)/2}([-1, 1] \times [0, t_f])$$

This solution is an analytic function of ε at the point $\varepsilon = 0$.

The uniqueness of the solution can be established by contradiction.

3.3. Remark regarding the exact solution of the linearized microconvection problem

We have examined the problem of the solvability of initial-boundary-value problems that arise during the procedure for finding the exact solution of the complete model of microconvection. The exact, invariant solution (2.1) was constructed for the linearized model of microconvection and for the Oberbeck–Boussinesq model of convection.[1,2] Periodic solutions were studied in the case of $U(t) = -\sin\gamma t$ (see problem (3.2)) and were investigated numerically.[14] For small values of ε and values of γ comparable with unity, a conclusion was drawn regarding the features of a microconvection regime, specifically regarding the helical periodic motion (in which the primary turn is an ellipse) of a fluid particle with a slow drift in the vertical direction. This conclusion was confirmed by an analysis of the non-trivial component of the motion based on the Krylov–Bogolyubov averaging method.[14] Unlike the complicated drift of a fluid particle predicted by the linearized model of microconvection, the trajectory calculated using the Oberbeck–Boussinesq model is a segment of a vertical straight line. Qualitative differences in the behaviour of trajectories obtained form the classical model of convection and the model of microconvection were thereby demonstrated.

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